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Existence and uniqueness theorems for boundary value problems involving incrementally non linear models

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Abstract

Sufficient conditions for existence and uniqueness of rate boundary value problem involving an incrementally non-linear constitutive equation useful for geomaterials are proved. The models considered are incrementally non-linear models not descending from a rate potential and thus are not included into the Hill theory framework. Results are given in the framework of small strain assumption. The main assumption is the positiveness of the second-order work. The existence theorem is a generalisation of the Lax-Milgram theorem, whereas uniqueness theorem is based on the well-known Hill exclusion functional. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Considerable work has been performed the last three decades to develop constitutive equations for materials. Many models have been advocated on the basis of comparison between experimental data and theoretical predictions, for elementary stress paths. These models are usually implemented in finite element codes. However, there is often a lack of firm mathematical results about the well posedness of the problem in the Hadamard sense (which means existence, uniqueness, and continuity of the solution with respect to the boundary conditions). For materials obeying a normality rule, Hill (1958, 1959, 1961, 1978) has formulated a rigorous theory of bifurcation and uniqueness. Following this pioneering work, Petryk (1989, 1991, 1992) has extended these results to other instability problems under rather weak assumptions, for instance, the incremental bilinearity (which means a loading and an unloading branch) is not necessary. The main assumption concerning the constitutive equations in Hill (1959, 1961) and Petryk (1989, 1991, 1992) is that it

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admits a velocity gradient potential. Restricted to an incrementally bilinear model, this assumption implies that the material obeys normality conditions. These constitutive equations are then unable to model the behaviour of geomaterials (soils, rocks and concrete), which exhibit a shear dilatancy usually not compatible with a normality rule. Other arguments concerning experimental evidence about isochoric loading paths can be used to prove clearly the necessity of non-normality for geomaterials (Darve and Chau, 1987; Nova, 1989). Mathematical general results concerning uniqueness for elastoplastic non-standard materials have been proved by Raniecki (1979) and Raniecki and Bruhns (1981). In the restricted case of small strains, it turns out that the result given by Raniecki implies that the positiveness of the second order work is a sufficient condition of uniqueness (Bigoni and Hueckel, 1991).

In order to properly model the occurrence of localisation in geomaterials, we have recently developed an incrementally non-linear model called CloE (Chambon, 1989; Chambon and Desrues, 1985; Chambon et al., 1994), which does not admit a velocity gradient potential. Some mathematical results have already been given for such models, but under very strong assumptions (Caillerie and Chambon, 1994). The objective of this paper is to prove that for such models and under small strain condition, the positiveness of the second-order work implies uniqueness of the solution of the rate boundary value problem. Under a slightly stronger assumption, existence is also proved. This paper is divided into four parts. The problems to be analysed are formulated in the first part. In the second, we prove a uniqueness theorem by using the Hill exclusion functional. The third part is devoted to the proof of a non-linear generalisation of the Lax–Milgram theorem (Lax and Milgram, 1954). Finally, in the fourth part, the link between second-order work and an existence theorem, based on the non-linear generalisation of the Lax–Milgram theorem is clarified.

As only a generic form of the model is used the presented results also hold true for hypoplastic models initially developed by Kolymbas (1987) and for a simple version of endochronic model (Bazant 1978).

2. The boundary value problem

2.1. The rate equilibrium problem

In the framework of small strain assumption, the rate equilibrium problem is:
find a velocity field $\dot{u}$ such that for any kinematically admissible virtual velocity yield $\dot{u}^*$

$$\int_{\Omega} \dot{\sigma}_{ij}\dot{\varepsilon}^*_{ij}\,\mathrm{d}\Omega - \int_{\Gamma_\sigma} \dot{t}_i\dot{u}^*_i\,\mathrm{d}\Gamma - \int_{\Omega} \dot{f}_i\dot{u}^*_i\,\mathrm{d}\Omega = 0, \tag{1}$$

and satisfying

$$\dot{u} = \dot{U} \quad \text{on} \quad \Gamma_{\dot{u}}, \tag{2}$$

where $\dot{\sigma}$ is the stress rate; $\dot{u}^*$ is a kinematically admissible virtual velocity field (derivable and vanishing on the part $\Gamma_{\dot{u}}$ of the boundary Γ of Ω where the velocities are prescribed); $\dot{\varepsilon}^*$ is the virtual strain rate, derived from the kinematically virtual velocity field:

$$\dot{\varepsilon}^*_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}^*_i}{\partial x_j} + \frac{\partial \dot{u}^*_j}{\partial x_i}\right), \tag{3}$$

where x_i is the space variable;

$\dot{t}$ is the traction rate prescribed on the part of the boundary Γ_σ such that $\Gamma_u \bigcup \Gamma_\sigma = \Gamma$ and $\Gamma_u \bigcap \Gamma_\sigma = \phi$; $\dot{f}$ is the rate of body forces per unit volume.

2.2. The constitutive models

We assume that the stress rate is related to the strain rate by the following constitutive equation:

$$\dot{\sigma} = A:(\dot{\varepsilon} + B\|\dot{\varepsilon}\|) = A:\dot{\varepsilon} + b\|\dot{\varepsilon}\|, \tag{4}$$

where A is a fourth order tensor, B and b are second-order tensors such that: $A{:}B = b$. These tensors depend on the state variables of the material under consideration. For any second-order tensor X, $\|X\|$ denotes the norm and is defined as:

$$\|X\|^2 = X:X = X_{ij}X_{ij} \tag{5}$$

In components eqn (4) writes:

$$\dot{\sigma}_{ij} = A_{ijkl}(\dot{\varepsilon}_{kl} + B_{kl}\|\dot{\varepsilon}\|) = A_{ijkl}\dot{\varepsilon}_{kl} + b_{ij}\|\dot{\varepsilon}\|. \tag{6}$$

CLoE models (Chambon, 1989; Chambon and Desrues, 1985; Chambon et al., 1994) are based on the basic eqn (4). In these models, only one-state variable is introduced, namely, the Cauchy stress σ. The dependence of tensors A and B on this variable is detailed in Chambon et al. (1994), for specific applications to sand. Many other possibilities have been explored (Chambon and Crochepeyre, 1995). Hypoplastic models (Kolymbas, 1987, 1993) can also be written in the form (4). For instance the first proposed hypoplastic model was written by Kolymbas (1987) as:

$$\dot{\sigma} = C_1(\sigma \cdot \dot{\varepsilon} + \dot{\varepsilon} \cdot \sigma) + C_2(\sigma : \dot{\varepsilon})\delta + \left(C_3\sigma + C_4\frac{\sigma \cdot \sigma}{\sigma : \sigma}\right)\|\dot{\varepsilon}\|, \tag{7}$$

where δ is the identity second order tensor. In components, eqn (7) reads:

$$\dot{\sigma}_{ij} = C_1(\sigma_{ik}\dot{\varepsilon}_{kj} + \dot{\varepsilon}_{il}\sigma_{lj}) + C_2\sigma_{kl}\dot{\varepsilon}_{kl}\delta_{ij} + \left(C_3\sigma_{ij} + C_4\frac{\sigma_{ik}\sigma_{kj}}{\sigma_{mn}\sigma_{mn}}\right)\|\dot{\varepsilon}\|, \tag{8}$$

from which it is therefore straightforward to deduce:

$$A_{ijkl} = C_1(\sigma_{ik}\delta_{lj} + \delta_{ik}\sigma_{lj}) + C_2\sigma_{kl}\delta_{ij}, \tag{9}$$

and

$$b_{ij} = C_3\sigma_{ij} + C_4\frac{\sigma_{ik}\sigma_{kj}}{\sigma_{mn}\sigma_{mn}}. \tag{10}$$

Many other proposed hypoplastic models can be written in the form of eqn (4). The simplest endochronic model (Bazant, 1978) can be written in this framework, except that the stress deviator s_{ij} and the strain rate deviator replace the stress and the strain rate, respectively. A simple generalisation of this model can be obtained, using eqn (4) with A_{ijkl} being the isotropic infinitesimal elasticity tensor and $b_{ij} = s_{ij}$.

Results presented in this paper apply straightforwardly to the quoted model of Kolymbas. The extension to the original model of Bazant (1978) should also not present difficulties, but is not pursued here.

3. Uniqueness theorem

3.1. Exclusion functional

Starting from eqn (1), it is straightforward (Hill 1958, 1978) to deduce that if

$$\int_{\Omega} \Delta\dot{\sigma}_{ij}\Delta\dot{\varepsilon}_{ij}\mathrm{d}\Omega > 0, \tag{11}$$

for every pair of distinct velocity fields $\dot{u}^1$ and $\dot{u}^2$, satisfying the boundary conditions (2), then uniqueness is guaranteed. Here, $\Delta\dot{\sigma}_{ij}$ and $\Delta\dot{\varepsilon}_{ij}$ stand for $\dot{\sigma}^2_{ij}-\dot{\sigma}^1_{ij}$ and $\dot{\varepsilon}^2_{ij}-\dot{\varepsilon}^1_{ij}$ and $\dot{\sigma}^2_{ij}$ and $\dot{\sigma}^1_{ij}$ are related respectively to $\dot{\varepsilon}^2_{ij}$ and $\dot{\varepsilon}^1_{ij}$ through the constitutive eqn (4).

It will be shown in the following that positiveness of second order work implies (11), and therefore uniqueness. To this purpose, we use a preliminary lemma.

3.2. *Lemma* 1

In the framework of models obeying eqn (4), if for every $\dot{\varepsilon}$ the second-order-work is positive, i.e. $\dot{\sigma}_{ij}\dot{\varepsilon}_{ij} \geqslant \alpha\|\dot{\varepsilon}\|^2$, with $\alpha \geqslant 0$ then for any pair of strain rates $\Delta\dot{\sigma}_{ij}\Delta\dot{\varepsilon}_{ij} \geqslant \alpha\|\Delta\dot{\varepsilon}\|^2$ and conversely.

Proof. The second order work $W(\dot{\varepsilon}) = \dot{\sigma}_{ij}\dot{\varepsilon}_{ij}$ satisfies

$$W(\dot{\varepsilon}) = \dot{\sigma}_{ij}\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}A_{ijkl}\dot{\varepsilon}_{kl}+\dot{\varepsilon}_{ij}b_{ij}\|\dot{\varepsilon}\| \geqslant \alpha\|\dot{\varepsilon}\|^2. \tag{12}$$

An application of eqn (12) to $\Delta\dot{\varepsilon}$ and $-\Delta\dot{\varepsilon}$ yields:

$$W(\Delta\dot{\varepsilon}) = \Delta\dot{\varepsilon}_{ij}A_{ijkl}\Delta\dot{\varepsilon}_{kl}+\Delta\dot{\varepsilon}_{ij}b_{ij}\|\Delta\dot{\varepsilon}\| \geqslant \alpha\|\Delta\dot{\varepsilon}\|^2, \tag{13}$$

$$W(-\Delta\dot{\varepsilon}) = \Delta\dot{\varepsilon}_{ij}A_{ijkl}\Delta\dot{\varepsilon}_{kl}+\Delta\dot{\varepsilon}_{ij}b_{ij}\|\Delta\dot{\varepsilon}\| \geqslant \alpha\|\Delta\dot{\varepsilon}\|^2, \tag{14}$$

We have therefore:

$$\Delta\dot{\sigma}_{ij}\Delta\dot{\varepsilon}_{ij} = \Delta\dot{\varepsilon}_{ij}A_{ijkl}\Delta\dot{\varepsilon}_{kl}+\Delta\dot{\varepsilon}_{ij}b_{ij}(\|\dot{\varepsilon}^2\|-\|\dot{\varepsilon}^1\|). \tag{15}$$

If $\Delta\dot{\varepsilon}_{ij}b_{ij} = 0$, then $W(\Delta\dot{\varepsilon}) = W(-\Delta\dot{\varepsilon}) = \Delta\dot{\sigma}_{ij}\Delta\dot{\varepsilon}_{ij}$, and the lemma is proved.

On the contrary, if $\Delta\dot{\varepsilon}_{ij}b_{ij} \neq 0$ starting from the triangular inequality:

$$|\|\dot{\varepsilon}^2\|-\|\dot{\varepsilon}^1\|| \leqslant \|\Delta\dot{\varepsilon}\|, \tag{16}$$

we have:

$$-\|\Delta\dot{\varepsilon}\| \leqslant \|\dot{\varepsilon}^2\|-\|\dot{\varepsilon}^1\| \leqslant \|\Delta\dot{\varepsilon}\|, \tag{17}$$

and then if $\Delta\dot{\varepsilon}_{ij}b_{ij} > 0$

$$-\|\Delta\dot{\varepsilon}\|\Delta\dot{\varepsilon}_{ij}b_{ij} \leqslant (\|\dot{\varepsilon}^2\| - \|\dot{\varepsilon}^1\|)\Delta\dot{\varepsilon}_{ij}b_{ij} \leqslant \|\Delta\dot{\varepsilon}\|\Delta\dot{\varepsilon}_{ij}b_{ij}, \tag{18}$$

which implies:

$$W(-\Delta\dot{\varepsilon}) \leqslant \Delta\dot{\sigma}_{ij}\Delta\dot{\varepsilon}_{ij} \leqslant W(\Delta\dot{\varepsilon}). \tag{19}$$

if $\Delta\dot{\varepsilon}_{ij}b_{ij} < 0$

$$\|\Delta\dot{\varepsilon}\|\Delta\dot{\varepsilon}_{ij}b_{ij} \leqslant (\|\dot{\varepsilon}^2\| - \|\dot{\varepsilon}^1\|)\Delta\dot{\varepsilon}_{ij}b_{ij} \leqslant -\|\Delta\dot{\varepsilon}\|\Delta\dot{\varepsilon}_{ij}b_{ij}, \tag{20}$$

then

$$W(\Delta\dot{\varepsilon}) \leqslant \Delta\dot{\sigma}_{ij}\Delta\dot{\varepsilon}_{ij} \leqslant W(-\Delta\dot{\varepsilon}). \tag{21}$$

Finally using eqns (14), (13), (19) and (21), it can be inferred that in any case:

$$\Delta\dot{\sigma}_{ij}\Delta\dot{\varepsilon}_{ij} \geqslant \alpha\|\Delta\dot{\varepsilon}\|^2. \tag{22}$$

Conversely assuming that for any pair of strain rates $\Delta\dot{\sigma}_{ij}\Delta\dot{\varepsilon}_{ij} \geqslant \alpha\|\Delta\dot{\varepsilon}\|^2$, taking $\dot{\varepsilon}^2_{ij} = 0$:

$$\forall\dot{\varepsilon}^1 \quad \Delta\dot{\sigma}_{ij}\Delta\dot{\varepsilon}_{ij} = \dot{\sigma}^1_{ij}\dot{\varepsilon}^1_{ij} \geqslant \alpha\|\Delta\dot{\varepsilon}\|^2 = \alpha\|\dot{\varepsilon}^1\|^2, \tag{23}$$

which concludes the proof.

We are in a position now to give the uniqueness theorem.

3.3. Uniqueness theorem

If in all points of the domain Ω and for all $\dot{\varepsilon}_{ij}$ the second-order work is positive $\dot{\sigma}_{ij}\dot{\varepsilon}_{ij} > 0$, then the solution of the rate boundary value problem defined in section 2.1., is unique.

Proof. According to the previous lemma, taking $\alpha = 0$, the positiveness of the second-order work implies

$$\Delta\dot{\sigma}_{ij}\Delta\dot{\varepsilon}_{ij} > 0, \tag{24}$$

then condition (11) is satisfied.

Comments. Finally, the positiveness of the second-order work in all points of a domain, is a sufficient condition of uniqueness for the corresponding rate boundary value problem involving hypoplastic constitutive equations. Such a condition can therefore be checked in finite element computations in order to estimate the confidence in the numerical results.

4. Lax–Milgram theorem for a non-linear problem

The Lax-Milgram theorem is the basic tool to prove existence for continuum and discrete linear problems (Dautray and Lions, 1988). An existence theorem for non-linear problems is the Browder–Minty theorem (Renardy and Rogers, 1993) or a similar theorem given by Lions (1969). The non linear problems studied here falls within the framework of the Browder–Minty theorem, the involved Hilbert space is a reflexive separable Banach space, and assuming eqns (47) and (48), the non-linear operator C_0 defined by eqn (35) is bounded, continuous, monotone, and coercive. However, we present here a less strong, but simpler theorem which is a straightforward generalisation of the Lax–Milgram theorem to a restricted class of non-linear problems which was given by Royis (1995) in a generic context.

4.1. Weak formulation of rate boundary value problem

From a mathematical point of view we can rephrase the problem described in section *2.1.* in a weak formulation as follows: find $\dot{u}$ such that

$$\dot{u} = \dot{U} \quad \text{on} \quad \Gamma_{\dot{u}}, \tag{25}$$

and

$$\forall \dot{u}^* \quad \text{such that} \quad \dot{u}^* = 0, \quad \text{on} \quad \Gamma_{\dot{u}}, \quad c(\dot{u}, \dot{u}^*) = L(\dot{u}^*), \tag{26}$$

where

$$c(\dot{u}, \dot{u}^*) = \int_\Omega \dot{\sigma}_{ij} \dot{\varepsilon}^*_{ij} \mathrm{d}\Omega = \int_\Omega (A_{ijkl} \dot{\varepsilon}_{kl} \dot{\varepsilon}^*_{ij} + b_{ij} \|\dot{\varepsilon}\| \dot{\varepsilon}^*_{ij}) \mathrm{d}\Omega, \tag{27}$$

and

$$L(\dot{u}^*) = \int_{\Gamma_\sigma} \dot{t}_i \dot{u}^*_i \mathrm{d}\Gamma + \int_\Omega \dot{f}_i \dot{u}^*_i \mathrm{d}\Omega. \tag{28}$$

This weak formulation results from eqns (1) and (4). The suitable functional space to analyze this problem is the Hilbert space $[H^1(\Omega)]^3$ (Dautray and Lions, 1988). Indeed, this gives sense to the integral involved in eqns (27) and (28). The precise formulation of the problem is therefore: find $\dot{u} \in v$ such that

$$\forall \dot{u}^* \in v_0, \quad c(\dot{u}, \dot{u}^*) = L(\dot{u}^*), \tag{29}$$

where:

$$v = \{ v \in [H^1(\Omega)]^3, v = \dot{U} \quad \text{on} \quad \Gamma_{\dot{u}} \}, \tag{30}$$

$$v_0 = \{ v \in [H^1(\Omega)]^3, v = 0 \quad \text{on} \quad \Gamma_{\dot{u}} \}, \tag{31}$$

where v_0 is a closed subspace of $[H^1(\Omega)]^3$, and then is a Hilbert space, and v is a closed submanifold of $[H^1(\Omega)]^3$. Following Korn's inequality (Duvaut and Lions, 1972, Dautray and Lions, 1988), it is known that:

$$\|\dot{u}^*\|_{v_0} = \left[\int_\Omega \|\dot{\varepsilon}^*\|^2 \mathrm{d}\Omega \right]^{\frac{1}{2}}, \tag{32}$$

is a norm for v_0 equivalent to the natural norm of $[H^1(\Omega)]^3$. This norm will be used in the following, the corresponding inner product being denoted by $\langle ., . \rangle_{v_0}$.

From eqn (27), it is clear that $c(\dot{u}, \dot{u}^*)$ is a bounded linear form on v_0 with respect to $\dot{u}^*$. Following the Riesz representation theorem, we can conclude that: $\exists C(\dot{u}) \in v_0$ such that:

$$\forall \dot{u}^* \in v_0, \quad < C(\dot{u}), \dot{u}^* > v_0 = c(\dot{u}, \dot{u}^*), \tag{33}$$

where $C(\dot{u})$ defines a non-linear mapping from v to v_0.

Similarly, from eqn (28), $L(\dot{u}^*)$ is a bounded linear form with respect to $\dot{u}^*$ and the Riesz representation theorem implies that: $\exists f \in v_0$ such that:

$$\forall \dot{u}^* \in v_0, \quad < f, \dot{u}^* > v_0 = L(\dot{u}^*). \tag{34}$$

Let $\dot{u}^0$ be any element of v, then $\dot{u}^{\triangleleft} = \dot{u} - \dot{u}^0$ belongs to v_0. Let us define

$$C_0(\dot{u}^{\triangleleft}) = C(\dot{u}^0 + \dot{u}^{\triangleleft}), \tag{35}$$

where $C_0(\dot{u}^{\triangleleft})$ is a mapping from v_0 to v_0.

The problem is now: find $\dot{u}^{\triangleleft} \in v_0$ such that:

$$C_0(\dot{u}^{\triangleleft}) = f, \tag{36}$$

or in an equivalent way: find $\dot{u}^{\triangleleft} \in v_0$ such that:

$$\dot{u}^{\triangleleft} = S(\dot{u}^{\triangleleft}), \tag{37}$$

where:

$$S(\dot{u}^{\triangleleft}) = \dot{u}^{\triangleleft} - \rho(C_0(\dot{u}^{\triangleleft}) - f), \tag{38}$$

with $\rho \neq 0$ and S is a mapping from v_0 to v_0.

4.2. *Lax–Milgram theorem in a non-linear case*

For incrementally non-linear problems, the Lax-Milgram theorem may be generalised as follows. If C_0 is continuous, i.e., $\exists M > 0$ such that for any pair of velocity fields belonging to the Hilbert space v_0, $\dot{u}^1$, $\dot{u}^2$:

$$\|C_0(\dot{u}^1) - C_0(\dot{u}^2)\|_{v_0}) \leqslant M\|\dot{u}^1 - \dot{u}^2\|_{v_0}, \tag{39}$$

C_0 is such that $\exists \alpha > 0$ such that for any pair of velocity fields belonging to the Hilbert space, v_0, $\dot{u}^1$, $\dot{u}^2$:

$$< \dot{u}^1 - \dot{u}^2, C_0(\dot{u}^1) - C_0(\dot{u}^2) > v_0 \geqslant \alpha\|\dot{u}^1 - \dot{u}^2\|^2_{v_0}, \tag{40}$$

then there exists one and only one solution for the fixed point problem given by eqn (38), and therefore for problems given by eqns (36) and (29).

Proof. Starting from eqn (38) we get

$$S(\dot{u}^1) - S(\dot{u}^2) = \dot{u}^1 - \dot{u}^2 - \rho(C(\dot{u}^1) - C(\dot{u}^2)), \tag{41}$$

which implies

$$\|S(\dot{u}^1) - S(\dot{u}^2)\|^2_{v_0} = \|\dot{u}^1 - \dot{u}^2\|^2_{v_0} - 2\rho < \dot{u}^1 - \dot{u}^2, C_0(\dot{u}^1) - C_0(\dot{u}^2) > v_0 + \rho^2\|C_0(\dot{u}^1) - C_0(\dot{u}^2)\|^2_{v_0}. \tag{42}$$

For $\rho > 0$, eqn (40) implies:

$$-\rho < \dot{u}^1 - \dot{u}^2, C_0(\dot{u}^1) - C_0(\dot{u}^2) > v_0 \leqslant -\rho\alpha\|\dot{u}^1 - \dot{u}^2\|^2_{v_0}, \tag{43}$$

and thus,

$$\|S(\dot{u}^1) - S(\dot{u}^2)\|^2_{v_0} \leqslant (1 - 2\rho\alpha + \rho^2 M^2)\|\dot{u}^1 - \dot{u}^2\|^2_{v_0}. \tag{44}$$

If we choose

$$\rho = \frac{\alpha}{M^2}, \tag{45}$$

which is strictly greater than zero, we obtain:

$$\|S(\dot{u}^1) - S(\dot{u}^2)\|^2_{v_0} \leqslant \left(1 - \frac{\alpha^2}{M^2}\right)\|\dot{u}^1 - \dot{u}^2\|^2_{v_0}. \tag{46}$$

As $1 - \alpha^2/M^2 < 1$, eqn (46) and the fixed point theorem imply the existence and uniqueness in the Hilbert space v_0.

5. Existence theorem

Now the generalised Lax–Milgram theorem given in section 4.2. is specified to the constitutive equations described in section 2.2.

5.1. Existence theorem

If $\forall \dot{\varepsilon}_{ij}$, $\exists M$ and $\exists \alpha > 0$ such that

$$\|\dot{\sigma}\| \leqslant M\|\dot{\varepsilon}\| \quad \forall P \in \Omega, \tag{47}$$

$$\dot{\sigma}_{ij}\dot{\varepsilon}_{ij} \geqslant \alpha\|\dot{\varepsilon}\|^2 \quad \forall P \in \Omega, \tag{48}$$

then there exists a solution in v of the rate boundary value problem defined in section 2.1. and involving a model obeying eqn (4).

The existence theorem follows from a preliminary lemma.

5.2. Lemma 2

If $\forall \dot{\varepsilon}_{ij}$ $\exists M$ and $\exists \alpha$ such that eqns (47) and (48) are satisfied then

$$\|\Delta\dot{\sigma}\| \leqslant M\|\Delta\dot{\varepsilon}\|, \tag{49}$$

$$\Delta\dot{\sigma}_{ij}\Delta\dot{\varepsilon}_{ij} \geqslant \alpha\|\Delta\dot{\varepsilon}\|^2, \tag{50}$$

for any pair of strain rates.

Proof of the lemma. Equation (50) follows from eqn (48) using lemma 3.2. Applying eqn (47) for $\Delta\dot{\varepsilon}_{ij}$ gives:

$$\|A_{ijkl}\Delta\dot{\varepsilon}_{kl}+b_{ij}\|\Delta\dot{\varepsilon}\|\| \leqslant M\|\Delta\dot{\varepsilon}\|. \tag{51}$$

We have therefore

$$\|\Delta\dot{\sigma}\|^2 = A_{ijkl}\Delta\dot{\varepsilon}_{kl}A_{ijmn}\Delta\dot{\varepsilon}_{mn}+b_{ij}b_{ij}(\|\dot{\varepsilon}^2\|-\|\dot{\varepsilon}^1\|)^2+2b_{ij}A_{ijkl}\Delta\dot{\varepsilon}_{kl}(\|\dot{\varepsilon}^2\|-\|\dot{\varepsilon}^1\|). \tag{52}$$

From eqn (16) we infer:

$$b_{ij}b_{ij}(\|\dot{\varepsilon}^2\|-\|\dot{\varepsilon}^1\|)^2 \leqslant b_{ij}b_{ij}\|\Delta\dot{\varepsilon}\|^2. \tag{53}$$

With a proper numeration of the strain rates, we can always assume that $b_{ij}A_{ijkl}\Delta\dot{\varepsilon}_{kl} \geqslant 0$ without changing the values of $\|\Delta\dot{\sigma}_{ij}\|$ and $\|\Delta\dot{\varepsilon}_{ij}\|$. Equation (16) implies therefore that:

$$2b_{ij}A_{ijkl}\Delta\dot{\varepsilon}_{kl}(\|\dot{\varepsilon}^2\|-\|\dot{\varepsilon}^1\|) \leqslant 2b_{ij}A_{ijkl}\Delta\dot{\varepsilon}_{kl}\|\Delta\dot{\varepsilon}\|, \tag{54}$$

and, finally, using inequalities (53) and (54) and eqn (52) yield:

$$\|\Delta\dot{\sigma}\|^2 \leqslant (\|A_{ijkl}\Delta\dot{\varepsilon}_{kl}+b_{ij}\|\Delta\dot{\varepsilon}\|\|)^2, \tag{55}$$

which terminates the proof.

5.3. *Proof of the existence theorem*

Let $\dot{u}^1$ and $\dot{u}^2$ be two fields belonging to v. Let us define $\dot{u}^{\lhd 1} = \dot{u}^1-\dot{u}^0$ and $\dot{u}^{\lhd 2} = \dot{u}^2-\dot{u}^0$. From eqns (27), (33) and (35) we have:

$$\forall \dot{u}^* \in v_0 \quad \langle C_0(\dot{u}^{\lhd 2})-C_0(\dot{u}^{\lhd 1}), \dot{u}^*\rangle v_0 = \langle C_0(\dot{u}^2)-C(\dot{u}^1), \dot{u}^*\rangle v_0$$
$$= \int_\Omega (\dot{\sigma}^2_{ij}-\dot{\sigma}^1_{ij})\dot{\varepsilon}^*_{ij}\mathrm{d}\Omega. \tag{56}$$

Now

$$\int_\Omega (\dot{\sigma}^2_{ij}-\dot{\sigma}^1_{ij})\dot{\varepsilon}^*_{ij}\mathrm{d}\Omega \leqslant \int_\Omega \|\dot{\sigma}^2-\dot{\sigma}^1\|\|\dot{\varepsilon}^*\|\mathrm{d}\Omega. \tag{57}$$

Inequality (49) and Cauchy inequality yield:

$$\int_\Omega (\dot{\sigma}^2_{ij}-\dot{\sigma}^1_{ij})\dot{\varepsilon}^*_{ij}\mathrm{d}\Omega \leqslant M\int_\Omega \|\dot{\varepsilon}^2-\dot{\varepsilon}^1\|\|\dot{\varepsilon}^*\|\mathrm{d}\Omega \leqslant M\|\dot{u}^2-\dot{u}^1\|_{v_0}\|\dot{u}^*\|_{v_0}, \tag{58}$$

therefore we have

$$\forall \dot{u}^* \in v_0 \quad \langle C_0(\dot{u}^{\lhd 2})-C_0(\dot{u}^{\lhd 1}), \dot{u}^*\rangle v_0 \leqslant M\|\dot{u}^2-\dot{u}^1\|_{v_0}\|\dot{u}^*\|_{v_0}. \tag{59}$$

Being $\dot{u}^2-\dot{u}^1 = \dot{u}^{\lhd 2}-\dot{u}^{\lhd 1}$, and taking $\dot{u}^* = C_0(\dot{u}^{\lhd 2})-C_0(\dot{u}^{\lhd 1})$ yields:

$$\|C_0(\dot{u}^{\lhd 2})-C_0(\dot{u}^{\lhd 1})\|^2_{v_0} \leqslant M\|\dot{u}^{\lhd 2}-\dot{u}^{\lhd 1}\|_{v_0}\|C_0(\dot{u}^{\lhd 2})C_0(\dot{u}^{\lhd 1})\|_{v_0}, \tag{60}$$

which implies inequality (39), assumption of the first non-linear version of the Lax–Milgram theorem.

Starting once more from eqns (27), (33), and (35) we have:

$$\langle C_0(\dot{u}^{\lhd 2})-C_0(\dot{u}^{\lhd 1}),\dot{u}^{\lhd 2}-\dot{u}^{\lhd 1}\rangle_{v_0}=\langle C(\dot{u}^2)-C(\dot{u}^1),\dot{u}^2-\dot{u}^1\rangle_{v_0}$$
$$=\int_\Omega(\dot{\sigma}_{ij}^2-\dot{\sigma}_{ij}^1)(\dot{\varepsilon}_{ij}^2-\dot{\varepsilon}_{ij}^1)\mathrm{d}\Omega. \quad (61)$$

Using inequality (50) yields:

$$\int_\Omega(\dot{\sigma}_{ij}^2-\dot{\sigma}_{ij}^1)(\dot{\varepsilon}_{ij}^2-\dot{\varepsilon}_{ij}^1)\mathrm{d}\Omega\geqslant\alpha\int_\Omega\|\dot{\varepsilon}^2-\dot{\varepsilon}^1\|^2\,\mathrm{d}\Omega, \quad (62)$$

which reads

$$\int_\Omega(\dot{\sigma}_{ij}^2-\dot{\sigma}_{ij}^1)(\dot{\varepsilon}_{ij}^2-\dot{\varepsilon}_{ij}^1)\mathrm{d}\Omega\geqslant\alpha\|\dot{u}^2-\dot{u}^1\|_{v_0}^2=\alpha\|\dot{u}^{\lhd 2}-\dot{u}^{\lhd 1}\|_{v_0}^2. \quad (63)$$

Inequality (63) finally implies inequality (40), second assumption of the non-linear Lax–Milgram theorem. Both assumptions of the Lax–Milgram theorem are therefore satisfied and consequently the existence theorem is proved.

6. Concluding remarks

Existence and uniqueness for a simple, non-linear constitutive model, useful in geomechanics have been analysed. The assumptions introduced for the existence theorem are stronger than the assumptions for the uniqueness theorem. This means that if existence of the solution is proved, then this solution is necessarily unique. The situation for CLoE like or hypoplastic models is then rather comparable to that of simpler constitutive equations, as for instance linear elasticity.

The result proved here is generally stronger than that previously presented (Caillerie and Chambon, 1994). Indeed, the conditions assumed in this earlier theorem have been proved to imply the positiveness of the second-order work. In some cases (tensor A_{ijkl} having six identical eigenvalues, for instance) it can be proved that the two conditions coincide, but such cases are not particularly interesting in view of applications. Therefore, the theorem presented in this paper is stronger than the previous.

The extension to the large strains may be difficult for the existence theorem, but seems easy for the uniqueness theorem.

The positiveness of the second-order work for all points of a domain has been proved to be sufficient condition of uniqueness of boundary value problems involving hypoplastic models.

This uniqueness theorem may be interesting in the sense that its basic assumption, the positiveness of the second-order work, is well known to be a sufficient condition for uniqueness for classical elastoplastic models, standard or non-standard, and for an incrementally non-linear model admitting a velocity gradient potential. This new result seems to indicate that the link between the positiveness of the second-order work and uniqueness is more fundamental and can likely be extended to other models.

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